Overview of the Multifractal Model of Asset Returns

Hye Woong Jeon

Abstract

We provide a mathematical overview of Benoit B. Mandelbrot's Multifractal Model of Asset Returns (MMAR) as described in his *The (Mis)Behavior of Markets*. We assume that the reader has an understanding of measure theory, Lebesgue integration, and measure-theoretic probability. Defining fractal dimension and scaling behavior as key themes, we first rigorously define fractal dimension. Then, we explore the fractal dimension of Brownian motion, and how fractal dimension is interlinked with scaling behavior. Finally, we conclude by constructing the MMAR.

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1 Fractal Dimension

Dimension offers a crude yet surprisingly informative description of size. Intuitively, dimension gives a rough estimate of how much "information" is contained in an object. For example, in the classic vector space definition of dimension, $\mathbb{R}^2$ has dimension 2 while $\mathbb{R}$ has dimension 1; one can see that $\mathbb{R}^2$ is capable of holding more information that $\mathbb{R}$.

In the same vein, essential to the idea behind fractal dimension is the concept of scaling behavior. Indeed, scaling behavior, fractial dimension, and their relationship occupy the main thread of this paper. Take a straight line L of length 100 centimeters i.e. 1 meter. We know beforehand that L has dimension 1. To see scaling behavior, we observe that measuring L requires a single meter-long stick, or 100 centimeter-long sticks. Furthermore, centimeter-long sticks are $\frac{1}{100}$th of the length of meter-long sticks. We can then generalize to the following equation

$$S = \lambda^{-D}, \text{ where } S = \text{number of sticks}, \lambda = \text{scaling factor, and } D = \text{dimension.}$$

The relationship can also be written as $D = -\log_\lambda S = -\frac{\log S}{\log \lambda}$. We check that $100 = \frac{1}{100}^{-1}$, as expected.

For the two-dimensional case, denote S_x as a square with side-length x-centimeters. We can cover a S_{100} with 16 S_{25}'s, or 100 S_1's. As expected, since S_{25} has a side-length that is $\frac{1}{4}$ of S_{100}'s, we have $16 = \frac{1}{4}^{-2}$. Similarly, $100 = \frac{1}{10}^{-2}$.

Dimension hence relies on observing how many "sticks" of a certain size are required to cover the object at hand, and how this quantity scales as the stick-size varies. This scaling concept also meshes well with the idea of dimension as an estimate of "information-holding capacity". With these ideas in mind, we introduce Hausdorff dimension as an important alternative to the restrictive vector space definition of dimension.

1.1 Hausdorff Dimension

Definition 1.1 (Diameter)**.** Let F be a nonempty subset of $\mathbb{R}^n$. The *diameter* of F, denoted by $|F|$, is defined as

$$|F| = \sup\{d(x,y) : x, y \in F\}, \text{ where } d(\cdot) \text{ is a metric on } \mathbb{R}^n.$$

Definition 1.2 (δ-cover)**.** Let $F \in \mathbb{R}^n$ and $\delta > 0$. Suppose $F \subset \bigcup_{i=1}^\infty U_i$, where for all i, $|U_i| \leq \delta$. Then $\{U_i\}_{i=1}^\infty$ is called a δ*-cover* of F. Furthermore, define $C_\delta(F)$ as the δ-covers of F.

Remark 1.3. Note that δ-covers are countable; this is possible because $\mathbb{R}^n$ is separable i.e. contains a countable dense subset.

Definition 1.4 (Hausdorff measure)**.** Let $s \geq 0$ and $F \subset \mathbb{R}^n$. Then for $\delta > 0$, define

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^\infty |U_i|^s : \{U_i\}_{i=1}^\infty \in C_\delta(F)\right\}.$$

That $\mathcal{H}^s_\delta$ exists follows from the fact that diameters are nonnegative. We then define the *s-dimensional Hausdorff measure* of F as

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

Proposition 1.5. *The s-dimensional Hausdorff measure is a measure.*

Proof. See [2]. □

Proposition 1.6 (Scaling property of Hausdorff measure)**.** *If $F \subset \mathbb{R}^n$ and $\lambda > 0$, then $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$, where $\lambda F = \{\lambda x : x \in F\}$.*

Proof. Let $\delta > 0$ and let $\{U_i\}$ be a δ-cover of F i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ where $|U_i| \leq \delta$ for all i. Notice that $\lambda F \subset \bigcup_{i=1}^{\infty} \lambda U_i$. Because $|\lambda U_i| = \lambda |U_i|$, it follows that

$$|\lambda U_i| = \lambda |U_i| \leq \lambda\delta, \text{ so } \{\lambda U_i\} \text{ is a } \lambda\delta\text{-cover of } \lambda F.$$

By definition, we have $\mathcal{H}^s_{\lambda\delta}(\lambda F) \leq \sum_{i=1}^{\infty} |\lambda U_i|^s = \lambda^s \sum_{i=1}^{\infty} |U_i|^s$. Given that this holds for any δ-cover $\{U_i\}$ of F, we have $\mathcal{H}^s_{\lambda\delta}(\lambda F) \leq \lambda^s \mathcal{H}^s_\delta(F)$. Taking $\delta \to 0$, we get $\mathcal{H}^s(\lambda F) \leq \lambda^s \mathcal{H}^s(F)$.

For the opposite inequality, it follows by similar logic that $\mathcal{H}^s_{\frac{\delta}{\lambda}}(F) \leq \left(\frac{1}{\lambda}\right)^s \mathcal{H}^s_\delta(\lambda F)$. Taking $\delta \to 0$, we get $\mathcal{H}^s(\lambda F) \geq \lambda^s \mathcal{H}^s(F)$, so $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$. □

Given $\delta < 1$ and $F \subset \mathbb{R}^n$, it follows by definition of Hausdorff measure that $\mathcal{H}^s(F)$ increases as s increases. Furthermore, if $t > s$ and $\{U_i\}$ is a δ-cover of F, then $\sum |U_i|^t \leq \delta^{t-s} \sum |U_i|^s$. Therefore, it follows by taking infima and letting $\delta \to 0$ that if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$. This is to say that there exists a p such that the Hausdorff measure "jumps" from ∞ to 0; this critical p is defined as the *Hausdorff dimension.*

Definition 1.7 (Hausdorff dimension)**.** Let $F \subset \mathbb{R}^n$. We define the *Hausdorff dimension* of F as $\dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$ such that

$$\mathcal{H}^s(F) = \begin{cases} 0 & \text{if } s > \dim_H F \\ \infty & \text{if } s < \dim_H F \end{cases}.$$

Note that $\dim_H F$ can be $0, \infty$, or between 0 and ∞.

Proposition 1.8. *Hausdorff dimension has the following properties:*

1. ***Monotonicity.*** *If $E \subset F$, then $\dim_H E \leq \dim_H F$.*
2. ***Countable stability.*** *Suppose $\{U_i\}_{i=1}^{\infty}$ is a countable collection of sets. Then $\dim_H \left(\bigcup_{i=1}^{\infty} U_i\right) = \sup_{i \geq 1} \dim_H U_i$.*

Proof. Monotonicity follows from the monotonicity of the Hausdorff measure, as given in Proposition 1.5. For countable stability, observe that for any k, $\dim_H U_k \leq \dim_H \left(\bigcup_{i=1}^{\infty} U_i\right)$ by monotonicity. Hence $\sup_{i \geq 1} \dim_H U_i \leq \dim_H \left(\bigcup_{i=1}^{\infty} U_i\right)$. For the other inequality, consider $t > \sup_{i \geq 1} \dim_H U_i$. Countable sub-additivity of the Hausdorff measure gives $\mathcal{H}^t \left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^t(U_i) = 0$. This therefore implies that

$\dim_H\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sup_{i\geq 1} \dim_H U_i$. If this is not the case, then there exists $\epsilon > 0$ with $\sup_{i\geq 1} \dim_H U_i + \epsilon < \dim_H\left(\bigcup_{i=1}^{\infty} U_i\right)$ and $\mathcal{H}^{\epsilon+\sup_{i\geq 1}\dim_H U_i}\left(\bigcup_{i=1}^{\infty} U_i\right) = 0$, implying that $\dim_H\left(\bigcup_{i=1}^{\infty} U_i\right)$ cannot be the infimum, which is a contradiction. $\square$

Next, we identify several properties of the Hausdorff dimension regarding images of a specific class of functions.

Definition 1.9 (α-Hölder continuity)**.** A function $f : [0, \infty) \to \mathbb{R}$ is said to be *α-Hölder continuous* if there exists $c > 0$ such that for any $x, y \in [0, \infty)$,

$$|f(x) - f(y)| \leq c|x - y|^{\alpha}.$$

Furthermore, f is said to be *locally α-Hölder continuous* at $x \geq 0$ if there exists $\epsilon > 0$ and $c > 0$ such that for any $y \in [0, \infty)$ with $|x - y| < \epsilon$,

$$|f(x) - f(y)| \leq c|x - y|^{\alpha}.$$

Proposition 1.10. *Suppose $f : [0, 1] \to \mathbb{R}^d$ is an α-Hölder continuous function. Define $a \wedge b := \min(a, b)$. The following are true:*

1. $\dim_H(\mathrm{Graph}_f[0, 1]) \leq 1 + (1 - \alpha)\left(d \wedge \frac{1}{a}\right)$*, where* $\mathrm{Graph}_f[0, 1] = \{(t, f(t)) : t \in [0, 1]\} \subset \mathbb{R}^{d+1}$*, and*
2. $\dim_H f(A) \leq \frac{\dim_H A}{\alpha}$ *for any* $A \subset [0, 1]$*.*

Proof of 1. Let $\epsilon > 0$. Because f is α-Hölder continuous, there exists $c > 0$ such that for all $x, y \in [0, 1]$ with $|x - y| \leq \epsilon$, $|f(x) - f(y)| \leq c\epsilon^{\alpha}$. Now, observe that at most $\lceil \frac{1}{\epsilon} \rceil$ intervals of length ϵ are needed to cover $[0, 1]$. Call these intervals $\{I_n\}$. Then the α-Hölder condition yields that the image of each I_n can be contained in a ball of radius $c\epsilon^{\alpha}$.

We make two observations that give an estimate of how many balls of diameter ϵ are needed to cover $f([0, 1])$:

1. Every ball of radius $c\epsilon^{\alpha}$ can be covered by a constant multiple of $\epsilon^{d\alpha-d}$ balls of diameter ϵ. This follows from the volume calculation
$$\frac{\Gamma_1 \epsilon^{\alpha\cdot d}}{\Gamma_2 \epsilon^d}, \text{ where } \Gamma_1, \Gamma_2 \text{ are the appropriate constants in the volume formula.}$$
2. Subintervals of length $\left(\frac{\epsilon}{c}\right)^{\frac{1}{\alpha}}$ map into balls of diameter ϵ. This means that a constant multiple of $\epsilon^{1-\frac{1}{\alpha}}$ balls of diameter ϵ are needed to cover a ball of radius $c\epsilon^{\alpha}$. This follows from the length calculation
$$\frac{\epsilon}{\left(\frac{\epsilon}{c}\right)^{\frac{1}{\alpha}}} = K\epsilon^{1-\frac{1}{\alpha}}, \text{ where } K \text{ is the appropriate constant.}$$

To cover $\mathrm{Graph}_f[0, 1]$, consider the products of intervals $\{I_n\}$ and balls of diameter ϵ. The first observation asserts that a constant multiple of $\epsilon^{d\alpha-d-1}$ product sets are necessary to cover $\mathrm{Graph}_f[0, 1]$, while the second observation posits instead that a constant multiple of $\epsilon^{-\frac{1}{\alpha}}$ product sets are necessary.

Notice that the diameters of the product sets are of order ϵ. Denote $M\epsilon$ as the greatest diameter among the product sets in either construction. For the first observation, if $s > -(d\alpha - d - 1)$, then $\mathcal{H}^s_{M\epsilon}(\text{Graph}_f[0,1]) = 0$ by definition of Hausdorff measure. The second observation yields that if $s > \frac{1}{\alpha}$, then $\mathcal{H}^s_{M\epsilon}(\text{Graph}_f[0,1]) = 0$. It then follows that

$$\dim_H(\text{Graph}_f[0,1]) \leq 1 + (1-\alpha)\left(d \wedge \frac{1}{\alpha}\right).$$

□

Proof of 2. By Proposition 1.6 and the α-Hölder condition, we have that

$$\mathcal{H}^{\frac{s}{\alpha}}(f(A)) \leq c^{\frac{s}{\alpha}}\mathcal{H}^s(A) \text{ for any } s \geq 0.$$

Hence if $s > \dim_H A$, then $\mathcal{H}^{\frac{s}{\alpha}}(f(A)) = 0$. It follows that for any $t > \frac{\dim_H A}{\alpha}$, $\mathcal{H}^t(f(A)) = 0$, so $\dim_H f(A) \leq \frac{\dim_H A}{\alpha}$. □

Corollary 1.11. *Proposition 1.10 holds even if f is locally α-Hölder continuous.*

Proof. This follows from the countable stability of the Hausdorff dimension. □

Remark 1.12. Other interpretations of fractal dimension exist, namely box-counting dimension (also known as Minkowski dimension). Box-counting dimension has several methods of calculation, the easiest being overlaying a grid with length δ squares onto a set, and counting how many squares intersect with that set - the dimension is then obtained by observing how this count scales as $\delta \to 0$. Although useful, we choose to omit discussions of other definitions for the sake of clarity.

2 Brownian Motion

We now turn our attention to random processes and investigate their fractal dimensions in order to discover a connection to scaling behavior.

Definition 2.1 (Gaussian random vector)**.** A vector $X = (X_1, ..., X_n)^T$ of random variables is called a *Gaussian random vector* if there exists a matrix $A \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^n$ such that

$$X = AY + b, \text{ where } Y \sim N(\mathbf{0}, I_m) \text{ and } I_m \text{ is the } m \times m \text{ identity matrix.}$$

Definition 2.2 (Gaussian random process)**.** Let T be an index set. A stochastic process $\{Y(t) : t \in T\}$ is called a *Gaussian random process* if for all $t_1 < \cdots < t_n$ with $t_1, ..., t_n \in T$, the vector $(Y(t_1), \cdots, Y(t_n))^T$ is a Gaussian random vector.

Definition 2.3 (Brownian motion in $\mathbb{R}^d$)**.** A stochastic process $\{B(t) : t \geq 0\}$ of R^d-valued random variables is called a *standard Brownian motion in* $\mathbb{R}^d$ if it satisfies the following:

1. $B(0) = \mathbf{0}$.
2. Has independent increments i.e. $B(t_n) - B(t_{n-1}) \perp \cdots \perp B(t_2) - B(t_1)$ for $0 \leq t_1 \leq \cdots \leq t_n$.

3. For all $t \geq 0$ and $h > 0$, $B(t+h) - B(t) \sim N(\mathbf{0}, hI_d)$, where I_d is the $d \times d$ identity matrix.

4. The sample function $t \mapsto B(t)$ is continuous almost surely.

Hereafter we refer to Brownian motion as BM, and standard Brownian motion as sBM, with no distinction between real or vector-valued BM.

We introduce several properties of sBM that prove useful in finding its fractal dimension. See [4] for the detailed proofs.

Proposition 2.4. *Let $\{B(t) : t \geq 0\}$ be a sBM, and $a > 0$. Then $\{X(t) : t \geq 0\}$ with $X(t) = \frac{1}{a}B(a^2 t)$ is also a sBM.*

Proposition 2.5. *Let $\{B(t) : t \geq 0\}$ be a sBM. Then $\{X(t) : t \geq 0\}$ with*

$$X(t) = \begin{cases} 0 & \text{if } t = 0 \\ tB\left(\frac{1}{t}\right) & \text{if } t > 0 \end{cases} \text{ is also a sBM.}$$

Theorem 2.6. *If $\alpha < \frac{1}{2}$, then Brownian motion is everywhere locally α-Hölder continuous.*

2.1 Calculation of fractal dimension

We first determine the upper bounds for the Hausdorff dimension of Brownian motion.

Theorem 2.7. *For any fixed $A \subset [0, \infty)$, the graph of d-dimensional Brownian motion $\{B(t) : t \geq 0\}$ satisfies*

$$\dim_H(\mathrm{Graph}_B A) \leq \begin{cases} \frac{3}{2} & \text{if } d = 1 \\ 2 & \text{if } d \geq 2 \end{cases} \text{ almost surely.}$$

Furthermore,

$$\dim_H B(A) \leq (2 \cdot \dim_H A) \wedge d \text{ almost surely.}$$

Proof. This follows from Theorem 2.6 and Corollary 1.11. □

It remains to find the lower bounds. For this, we introduce the potential theoretic method.

Definition 2.8 (α-potential, α-energy)**.** Suppose μ is a mass distribution on a metric space (E, ρ) and $\alpha \geq 0$. Then *α-potential* of a point $x \in E$ with respect to μ is defined as

$$\phi_\alpha(x) = \int \frac{1}{\rho(x,y)^\alpha} d\mu(y).$$

The *α-energy* of μ is then defined as

$$I_\alpha(\mu) = \int \phi_\alpha(x) d\mu(x) = \iint \frac{1}{\rho(x,y)^\alpha} d\mu(x) d\mu(y).$$

Theorem 2.9 (Potential theoretic method)**.** *Let $\alpha \geq 0$ and μ be a mass distribution on a metric space (E, ρ). Then for every $\epsilon > 0$,*

$$\mathcal{H}^{\alpha}_{\epsilon}(E) \geq \frac{\mu(E)^2}{\iint_{\rho(x,y)<\epsilon} \frac{1}{\rho(x,y)^{\alpha}} d\mu(x) d\mu(y)}.$$

Therefore, if $I_\alpha(\mu) < \infty$, then $\mathcal{H}^{\alpha}(E) = \infty$, so $\dim_H E \geq \alpha$.

Proof. Let $\epsilon > 0$ and $\delta > 0$. By definition of Hausdorff measure, there exists a pairwise disjoint covering $\{A_n\}$ of E with each set having diameter less than ϵ such that

$$\sum_{n=1}^{\infty} |A_n|^{\alpha} \leq \mathcal{H}^{\alpha}_{\epsilon}(E) + \delta.$$

We have

$$\iint_{\rho(x,y)<\epsilon} \frac{1}{\rho(x,y)^{\alpha}} d\mu(x) d\mu(y) \geq \sum_{n=1}^{\infty} \iint_{A_n \times A_n} \frac{1}{\rho(x,y)^{\alpha}} d\mu(x) d\mu(y).$$

For $y \in A_n$, it follows that $\rho(x,y)^{\alpha} \leq |A_n|^{\alpha}$ for any $x \in A_n$. Hence $\frac{1}{|A_n|^{\alpha}} \leq \frac{1}{\rho(x,y)^{\alpha}}$. It then follows that

$$\sum_{n=1}^{\infty} \iint_{A_n \times A_n} \frac{1}{\rho(x,y)^{\alpha}} d\mu(x) d\mu(y) \geq \sum_{n=1}^{\infty} \int_{A_n} \frac{\mu(A_n)}{|A_n|^{\alpha}} d\mu(y) \geq \sum_{n=1}^{\infty} \frac{\mu(A_n)^2}{|A_n|^{\alpha}}.$$

Furthermore, by countable sub-additivity, we have

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} |A_n|^{\frac{\alpha}{2}} \frac{\mu(A_n)}{|A_n|^{\frac{\alpha}{2}}}.$$

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\mu(E)^2 &\leq \sum_{n=1}^{\infty} |A_n|^{\alpha} \cdot \sum_{n=1}^{\infty} \frac{\mu(A_n)^2}{|A_n|^{\alpha}} \\
&\leq (\mathcal{H}^{\alpha}_{\epsilon}(E) + \delta) \cdot \sum_{n=1}^{\infty} \frac{\mu(A_n)^2}{|A_n|^{\alpha}} \\
&\leq (\mathcal{H}^{\alpha}_{\epsilon}(E) + \delta) \cdot \iint_{\rho(x,y)<\epsilon} \frac{1}{\rho(x,y)^{\alpha}} d\mu(x) d\mu(y).
\end{aligned}$$

Since this inequality holds for all $\delta > 0$, it follows that as $\delta \to 0$,

$$\mathcal{H}^{\alpha}_{\epsilon}(E) \geq \frac{\mu(E)^2}{\iint_{\rho(x,y)<\epsilon} \frac{1}{\rho(x,y)^{\alpha}} d\mu(x) d\mu(y)}.$$

Furthermore, if $I_\alpha(\mu) < \infty$, then as $\epsilon \to 0$, $\mathcal{H}^{\alpha}_{\epsilon}(E) \to \infty$. Hence $\mathcal{H}^{\alpha}(E) = \infty$, so $\dim_H E \geq \alpha$. □

Remark 2.10. To conclude that $\dim_H E \geq \alpha$ almost surely, it suffices to have $\mathbb{E}(I_\alpha(\mu)) < \infty$. This follows from Markov's inequality. Keeping in mind that $I_\alpha(\mu)$ has a dependency on a metric space (E, ρ) for a random metric space (E, ρ), we have

$$\epsilon \cdot \mu(\{E : I_\alpha(\mu) \geq \epsilon\}) \leq \mathbb{E}(I_\alpha(\mu)) < \infty.$$

Since this holds for all $\epsilon > 0$, it follows that $I_\alpha(\mu) < \infty$ almost surely.

Theorem 2.11. *Suppose $\{B(t) : t \in [0,1]\}$ is a d-dimensional BM.*

1. *If $d = 1$, then $\dim_H(\text{Graph}_B[0,1]) = \frac{3}{2}$ almost surely.*

2. *If $d \geq 2$, then $\dim_H B([0,1]) = \dim_H(\text{Graph}_B[0,1]) = 2$ almost surely.*

Proof of 1. Define a measure μ on $\text{Graph}_B[0,1]$, where for any Borel $A \subset [0,1] \times \mathbb{R}$,

$$\mu(A) = \mathcal{L}(t \in [0,1] : (t, B(t)) \in A), \text{ with Lebesgue measure } \mathcal{L}.$$

Let $0 < \alpha < \frac{3}{2}$. By changing variables, we have

$$\begin{aligned}
\mathbb{E}(I_\alpha(\mu)) &= \mathbb{E} \iint_{[0,1]\times\mathbb{R}} \frac{1}{|x-y|^\alpha} d\mu(x) d\mu(y) \\
&= \mathbb{E} \int_0^1 \int_0^1 \frac{1}{|(t, B(t)) - (s, B(s))|^\alpha} ds\ dt \\
&= \mathbb{E} \int_0^1 \int_0^1 \frac{1}{|(t-s)^2 + (B(t) - B(s))^2|^{\frac{\alpha}{2}}} ds\ dt.
\end{aligned}$$

By Fubini's theorem, it follows that

$$\mathbb{E}(I_\alpha(\mu)) \leq 2 \int_0^1 \mathbb{E}((t^2 + B(t)^2)^{-\frac{\alpha}{2}}) dt.$$

Notice that $B(t) = \sqrt{t} Z$, where $Z \sim N(0,1)$. Denoting $z \mapsto \mathfrak{p}(z)$ as the density function of $N(0,1)$, observe that $(t^2 + tz^2)^{-\frac{\alpha}{2}} \mathfrak{p}(z) = (t^2 + t(-z)^2)^{-\frac{\alpha}{2}} \mathfrak{p}(-z)$ for any nonnegative $z \in \mathbb{R}$. Hence, we have

$$\begin{aligned}
\mathbb{E}((t^2 + B(t)^2)^{-\frac{\alpha}{2}}) &= \int_{-\infty}^{\infty} (t^2 + tz^2)^{-\frac{\alpha}{2}} \mathfrak{p}(z) dz \\
&= 2 \int_0^{\infty} (t^2 + tz^2)^{-\frac{\alpha}{2}} \mathfrak{p}(z) dz.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\int_0^\infty (t^2 + tz^2)^{-\frac{\alpha}{2}} \mathfrak{p}(z) dz &\leq \int_0^{\sqrt{t}} t^{-\alpha} dz + \int_{\sqrt{t}}^\infty (tz^2)^{-\frac{\alpha}{2}} \mathfrak{p}(z) dz \\
&= t^{\frac{1}{2}-\alpha} + t^{-\frac{\alpha}{2}} \int_{\sqrt{t}}^\infty z^{-\alpha} \mathfrak{p}(z) dz \\
&= t^{\frac{1}{2}-\alpha} + t^{-\frac{\alpha}{2}} \left(\int_{\sqrt{t}}^1 z^{-\alpha} \mathfrak{p}(z) dz + \int_1^\infty z^{-\alpha} \mathfrak{p}(z) dz \right) \\
&\leq t^{\frac{1}{2}-\alpha} + t^{-\frac{\alpha}{2}} \left(\int_{\sqrt{t}}^1 z^{-\alpha} dz + 1 \right).
\end{aligned}$$

In total, we obtain the following inequality:

$$\begin{aligned}\mathbb{E}(I_\alpha(\mu)) &\leq 4\left(\int_0^1 t^{\frac{1}{2}-\alpha} + t^{-\frac{\alpha}{2}}\left(\int_{\sqrt{t}}^1 z^{-\alpha}dz + 1\right)\right)\\ &= 4\left(\int_0^1 t^{\frac{1}{2}-\alpha} + t^{-\frac{\alpha}{2}} + \frac{t^{-\frac{\alpha}{2}} - t^{\frac{1}{2}-\alpha}}{1-\alpha}\right)\\ &= 4\left(\left.\frac{t^{\frac{3}{2}-\alpha}}{\frac{3}{2}-\alpha}\right|_0^1 + (\cdots)\right).\end{aligned}$$

Notice that the above is finite if $0 < \alpha < \frac{3}{2}$. It then follows from Theorem 2.9 that $\dim_H(\text{Graph}_B[0,1]) \geq \frac{3}{2}$. Combined with Theorem 2.7, we have our result.

□

Proof of 2. Define a measure μ on $B([0,1])$, where for any Borel set $A \subset \mathbb{R}^d$,

$$\mu(A) = \mathcal{L}(B^{-1}(A) \cap [0,1]), \text{ with Lebesgue measure } \mathcal{L}.$$

Let $0 < \alpha < 2$. Noting Remark 2.10 and change of variables, we want to show that

$$\mathbb{E}\iint_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} d\mu(x)d\mu(y) = \mathbb{E}\int_0^1\int_0^1 \frac{1}{|B(t)-B(s)|^\alpha} ds\ dt < \infty.$$

Notice that $|B(t)-B(s)|$ and $|t-s|^{-\frac{1}{2}}|B(1)|$ have the same distribution. Given that $z \mapsto \mathfrak{p}(z)$ is the density function of $N(\mathbf{0}, I_d)$, we have

$$\begin{aligned}\mathbb{E}(|B(t)-B(s)|^{-\alpha}) &= |t-s|^{-\frac{\alpha}{2}} \cdot \mathbb{E}(|B(1)|^{-\alpha})\\ &= |t-s|^{-\frac{\alpha}{2}} \cdot \underbrace{\int_{\mathbb{R}^d} |z|^{-\alpha}\mathfrak{p}(z)\ dz}_{<\infty}.\end{aligned}$$

By Fubini's theorem, it follows that

$$\begin{aligned}\mathbb{E}\int_0^1\int_0^1 \frac{1}{|B(t)-B(s)|^\alpha} ds\ dt &= \int_0^1\int_0^1 \mathbb{E}(|B(t)-B(s)|^{-\alpha})\ ds\ dt\\ &= \left(\int_{\mathbb{R}^d} |z|^{-\alpha}\mathfrak{p}(z)\ dz\right) \cdot \int_0^1\int_0^1 |t-s|^{-\frac{\alpha}{2}} ds\ dt\\ &\leq \left(\int_{\mathbb{R}^d} |z|^{-\alpha}\mathfrak{p}(z)\ dz\right) \cdot 2\int_0^1 u^{-\frac{\alpha}{2}} du < \infty.\end{aligned}$$

Remark 2.10 and Theorem 2.9 gives that $\dim_H B([0,1]) \geq \alpha$ almost surely, and the result follows by taking $\alpha \to 2$. Now, observe that $B([0,1])$ is simply a projection of $\text{Graph}_B[0,1]$. Given the compactness of $[0,1]$ and the almost-sure continuity of B, it follows that $\dim_H(\text{Graph}_B[0,1]) \geq 2$ almost surely also.

Finally, in conjunction with Theorem 2.7, we have our result. □

3 Fractional Brownian Motion

Definition 3.1 (Fractional Brownian motion)**.** A stochastic process $\{B_H(t) : t \geq 0\}$ is called a *fractional Brownian motion with Hurst index* $H \in (0,1)$ if it is a centered Gaussian process with covariance function

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

We denote a fractional Brownian motion with Hurst index H as fBM_H.

Remark 3.2. The mean and covariance function suffice to define a Gaussian process, as the Gaussian distribution's characteristic function depends only on its mean and covariance matrix.

Proposition 3.3. $\text{fBM}_{\frac{1}{2}}$ *is equivalent to sBM.*

Proof. This follows from calculating the covariance function of $\text{fBM}_{\frac{1}{2}}$. □

Proposition 3.4. fBM_H *has stationary increments. Furthermore, increments are dependent, with the nature of the dependence relying on* H*.*

Proof. We skip the proof of stationary increments - the curious reader can find a proof in [5]. To show that increments are dependent, consider $s_1, t_1, s_2, t_2 \in \mathbb{R}$ such that $s_1 < t_1 < s_2 < t_2$. Then the covariance function yields

$$\mathbb{E}[(B_H(t_1) - B_H(s_1))(B_H(t_2) - B_H(s_2))] = \frac{1}{2}\left((t_2 - s_1)^{2H} - (t_2 - t_1)^{2H} - (s_2 - s_1)^{2H} + (s_2 - t_1)^{2H}\right).$$

The function $x \mapsto x^{2H}$ is concave if $H < \frac{1}{2}$, and convex if $H > \frac{1}{2}$. Therefore, we have

$$\mathbb{E}[(B_H(t_1) - B_H(s_1))(B_H(t_2) - B_H(s_2))] \begin{cases} < 0 & \text{if } H < \frac{1}{2} \\ > 0 & \text{if } H > \frac{1}{2} \end{cases}.$$

More specifically, if $H < \frac{1}{2}$, trends are less likely to persist i.e. *antipersistent*, while $H > \frac{1}{2}$ means that trends are likely to continue i.e. *persistent*. □

Proposition 3.5. *Let* $\{B_H(t) : t \geq 0\}$ *be a* fBM_H*, and* $a > 0$*. Then* $\{B_H(at)\}$ *is equivalent to* $\{a^H B_H(t)\}$*. Therefore,* fBM *displays scaling behavior based on the Hurst index.*

Proof. This follows from a straightforward calculation of the covariances of both processes. □

Theorem 3.6. *Suppose* $\{B_H(t) : t \geq 0\}$ *is a* fBM_H*. Then* $\dim_H(\text{Graph}_{B_H}[0,1]) = 2 - H$*.*

Proof. This follows by similar logic to Theorem 2.11. □

Remark 3.7. This gives a glimpse of the critical relationship between fractal dimension and the Hurst index. Furthermore, in conjunction with Proposition 3.5, we see how fractal dimension behavior interacts with fBM's scaling behavior. See Figure 1 for a visual example.

4 Multifractal Model of Asset Returns (MMAR)

Our discussions about fractal dimension and scaling laws culminate in Mandelbrot's Multifractal Model of Asset Returns. Mandelbrot identifies two problematic assumptions in contemporary financial models: thin tails (hence finite variance) and independence of separate time periods.

The independence of separate time periods can be solved with fBM, as we have control over trends by adjusting the Hurst index. The former problem can be solved with a fractal conception of time. In the market, trading activity is not uniform across time; instead, there are bursts of activity interspersed between periods of calm. Constructing a fractal measure μ of a finite time interval $[0, T]$, then taking the c.d.f. of μ gives a very flexible model for trading activity.

We first define multifractal processes. Observations of multifractal behavior will be made in later sections.

Definition 4.1 (Multifractal process)**.** A stochastic process $\{X(t) : t \geq 0\}$ is called multifractal if it has stationary increments and satisfies for all $t \in \mathcal{T}, q \in \mathcal{Q}$

$$\mathbb{E}[|X(t + \Delta t) - X(t)|^q] \sim c(q)(\Delta t)^{\tau(q)+1} \text{ as } \Delta t \to 0,$$

where $\mathcal{T}, \mathcal{Q} \subset \mathbb{R}$ are intervals with positive lengths, $0 \in \mathcal{T}$, and $[0, 1] \subset \mathcal{Q}$.[1] Furthermore, $c, \tau : \mathcal{Q} \to \mathbb{R}$. We call τ the *scaling function* of $\{X(t)\}$.

We justify the use of absolute moments in the definition of multifractality. Recall that fractal dimension has an intimate relationship with scaling behavior. In the same vein, multifractal processes are similarly defined by scaling properties on the process's absolute moments.

Suppose $\{X(t)\}$ is a multifractal process. Because $\{X(t)\}$ has stationary increments, $X(c+t) - X(t)$ has the same distribution as $X(t)$. Hence we can equate t with time difference. Rewrite the qth absolute moment as

$$\mathbb{E}[|X(t)|^q] = \mathbb{E}[|X(t)|^{q-1} \cdot |X(t)|].$$

In this way, the qth absolute moment can be viewed as a weighted average, "summed" over components with the form

$$\underbrace{|X(t)|^{q-1} \cdot \{\text{probability of } X(t)\}}_{\text{weight}} \cdot \underbrace{|X(t)|}_{\text{value}}.$$

The qth absolute moment can thus be seen as describing behavior far away from the mean. Now, we compare $\mathbb{E}[|X(t)|^q]$ with $\mathbb{E}[|X\left(\frac{t}{2}\right)|^q]$:

$$\begin{aligned} \mathbb{E}\left[\left|X\left(\frac{t}{2}\right)\right|^q\right] &\sim c(q) \cdot t^{\tau(q)+1} \cdot \left(\frac{1}{2}\right)^{\tau(q)+1} \\ &\sim \mathbb{E}[|X(t)|^q] \cdot \left(\frac{1}{2}\right)^{\tau(q)+1}. \end{aligned}$$

[1] We write that $f \sim g$ if $\frac{f(x)}{g(x)} \to 1$ as $x \to 0$.

By halving the time difference, we see that the behavior far away from the mean scales appropriately, as expected.

4.1 Construction of multifractal measures

For a simple exercise, consider the binomial multiplicative cascade. Let $X = [0,1]$, and let $m_0, m_1 \in (0,1)$ such that $m_0 + m_1 = 1$. By halving X and redistributing the mass according to m_0 and m_1, we obtain the measure μ_1, where

$$\mu_1([0,1/2]) = m_0 \text{ and } \mu_1([1/2,1]) = m_1.$$

Further halving $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ and redistributing gives the measure μ_2, where

$$\begin{aligned}
&\mu_2[0,1/4] = m_0^2, \\
&\mu_2[1/4,1/2] = \mu_2[1/2,3/4] = m_0 m_1, \text{ and} \\
&\mu_2[3/4,1] = m_1^2.
\end{aligned}$$

Continuing this recursive halving and redistribution process, we obtain the binomial measure μ. Formally, μ is the limit of the iterated measures. We can further generalize the binomial measure; instead of halving, we can divide any interval into $b > 2$ sub-intervals and redistribute according to weights $m_0, ..., m_{b-1}$, which sum to 1. Even further, the weights assigned to each sub-interval can be randomized. See Figure 2 for a concrete visualization.

Definition 4.2 (Microcanonical multiplicative cascade)**.** Let $X = [0,1]$, and let $b > 2$. The *microcanonical multiplicative cascade* is constructed as follows:

1. Subdivide X into b equal intervals. Call these sub-intervals $X_0, ..., X_{b-1}$.
2. Suppose $M_0, ..., M_{b-1}$ are identically distributed from a distribution M such that $\sum_{i=0}^{b-1} M_i = 1$. Call these random variables *multipliers.*
3. Define the measure μ_1, where $\mu_1(X_\beta) = M_\beta$ for $0 \le \beta < b$.
4. For $0 \le \beta < b$, subdivide X_β into b equal intervals, and call these sub-intervals $X_{\beta 0}, ..., X_{\beta(b-1)}$.
5. For $0 \le \beta < b$, suppose $M_{\beta 0}, ..., M_{\beta(b-1)}$ are identically distributed from a distribution M such that $\sum_{i=0}^{b-1} M_{\beta i} = 1$. Furthermore, assume that for all $0 \le \beta < b$, $M_{\beta 0}, ..., M_{\beta(b-1)}$ are independent from $M_0, ..., M_{b-1}$.[2]
6. Define the measure μ_2, where $\mu_2(X_{\beta\gamma}) = M_{\beta\gamma}$ for $0 \le \beta, \gamma < b$.
7. Continue the above steps of subdivision, sampling, and redistribution.

Definition 4.3 (Canonical multiplicative cascade)**.** The *canonical multiplicative cascade* is constructed in the same manner as the microcanonical multiplicative cascade, but with one change: the multipliers on each stage should sum to 1 on average i.e. $\mathbb{E}\left(\sum_{i=0}^{b-1} M_i\right) = 1$. We call the random variable $\Omega = \sum_{i=1}^{b-1} M_i$ the *aggregate* of the canonical multiplcative cascade.

[2]In other words, multipliers at the same "stage" are identically distributed, and multipliers on different stages are independent from each other.

Proposition 4.4. *Let $X = [0,1]$, and $b > 2$. Choose M as the multiplier random variable, to which every multiplier is equal in distribution. Define μ_{mc} as a microcanonical multiplicative measure on X, and μ_c as a canonical multiplicative measure. If $t = \sum_{i=1}^{k} n_i b^{-i}$ where $n_1, ..., n_k \in \{0,1\}$, and $\Delta t = b^{-k}$, then for $q \geq 0$*

$$\mathbb{E}(\mu_{mc}([t, t+\Delta t])^q) = (\Delta t)^{\tau(q)+1}, \text{ and}$$
$$\mathbb{E}(\mu_c([t, t+\Delta t])^q) = \mathbb{E}(\Omega^q)(\Delta t)^{\tau(q)+1},$$

where $\tau(q) = -\log_b \mathbb{E}(M^q) - 1$.

Proof. Using the multiplier notation of Definition 4.2, notice that by construction of μ_{mc},

$$\mu_{mc}([t, t+\Delta t]) = M_{n_1} M_{n_1 n_2} \cdots M_{n_1 n_2 \ldots n_k}.$$

Since multipliers are identically and independently distributed across stages, it then follows that

$$\begin{aligned}
\mathbb{E}(\mu_{mc}([t, t+\Delta t])^q) &= \mathbb{E}(M_{n_1}^q) \cdots \mathbb{E}(M_{n_1 \ldots n_k}^q) \\
&= \mathbb{E}(M^q)^k \\
&= b^{k \log_b \mathbb{E}(M^q)} \\
&= (b^{-k})^{-\log_b \mathbb{E}(M^q) - 1 + 1} = (\Delta t)^{\tau(q)+1}.
\end{aligned}$$

Similarly, by construction of μ_c,

$$\mu_c([t, t+\Delta t]) = \Omega \cdot M_{n_1} M_{n_1 n_2} \cdots M_{n_1 n_2 \ldots n_k}.$$

By similar logic, we get $\mathbb{E}(\mu_c([t, t+\Delta t])^q) = \mathbb{E}(\Omega^q)(\Delta t)^{\tau(q)+1}$. □

Remark 4.5. For our definitions of microcanonical and canonical measures, Proposition 4.4 holds only for intervals of size b^{-k}. This is to say that we are dealing with a discrete notion of time. It is, however, possible to construct continuous multiplicative cascades.

Proposition 4.4 gives that microcanonical and canonical multiplicative cascades display multifractal behavior in a similar manner to multifractal processes. In particular, we see scaling behavior in the absolute moments as a function of an interval's length. Hence, we arrive at the definition of multifractal measure.

Definition 4.6. Let $X \subset \mathbb{R}$ be an interval of the form $[0, T]$, where $0 < T \leq \infty$. A random measure μ defined on X is called a *multifractal measure* if for all $t \in X, q \in \mathcal{Q}$,

$$\mathbb{E}[\mu[t, t+\Delta t]^q] \sim c(q) \cdot (\Delta t)^{\tau(q)+1} \text{ as } \Delta t \to 0,$$

where $\mathcal{Q} \subset \mathbb{R}$ is an interval containing $[0,1]$, and $c, \tau : \mathcal{Q} \to \mathbb{R}$.

4.2 Model definition

Definition 4.7 (Compound process)**.** Suppose $\{X(t) : t \in [0, T]\}$, where $0 < T < \infty$, is defined as

$$X(t) = B_H(\theta(t)), \text{ where } \theta(t) = \mu([0, t]) \text{ and } \mu \text{ is a multifractal measure on } [0, T].$$

Furthermore, $\{B_H(t)\}$ and $\{\theta(t)\}$ are independent. The process $\{X(t)\}$ is called a *compound process*, and it describes the *multifractal model of asset returns*, where $X(t)$ represents the log-returns of a financial asset.

Theorem 4.8. *Let $0 < T < \infty$. The compound process $\{X(t) : t \in [0, T]\}$ is multifractal.*

Proof. Let $q \geq 0$. We want to show that $\mathbb{E}[|X(t)|^q]$ is a scaled power function. By the law of total expectation, we have that

$$\mathbb{E}[|X(t)|^q] = \mathbb{E}[|B_H(\theta(t))|^q] = \mathbb{E}[\mathbb{E}[|B_H(\theta(t))|^q \mid \theta(t) = u]].$$

Therefore, we calculate

$$\begin{aligned} \mathbb{E}[|B_H(\theta(t))|^q \mid \theta(t) = u] &= \mathbb{E}[|B_H(u)|^q \mid \theta(t) = u] \\ &= \theta(t)^{Hq} \cdot \mathbb{E}[|B_H(1)|^q]. \end{aligned}$$

Since $\{B_H(t)\}$ and $\{\theta(t)\}$ are independent, it follows that

$$\mathbb{E}[\mathbb{E}[|B_H(\theta(t))|^q \mid \theta(t) = u]] = \mathbb{E}[\theta(t)^{Hq}] \cdot \mathbb{E}[|B_H(1)|^q].$$

The multifractality of $\{\theta(t)\}$ gives that

$$\mathbb{E}[\theta(t)^{Hq}] \sim c_\theta(Hq) t^{\tau_\theta(Hq)+1} \text{ as } t \to 0.$$

Define $c_X(q) = c_\theta(Hq) \cdot \mathbb{E}[|B_H(1)|^q]$ and $\tau_X(q) = \tau_\theta(Hq)$. It then follows that

$$\mathbb{E}[|X(t)|^q] \sim c_X(q) t^{\tau_X(q)+1} \text{ as } t \to 0, \text{ so } \{X(t)\} \text{ is multifractal.}$$

□

Remark 4.9. It can be shown that the compound process generates a wide degree of tail behaviors. In particular, using a microcanonical multiplicative cascade as the basis for $\theta(t)$ generates thin tails, while a canonical multiplicative cascade generates much fatter tails. Hence the compound process allows for a great degree of flexibility while maintaining simplicity.

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A Supplementary material

A.1 Properties of multifractal processes

Let $\{X(t)\}$ be a multifractal process with scaling function τ.

Theorem A.1. *For a bounded interval* $[0, T] \supset [0, 1]$*, the scaling function satisfies the following:*

1. $\tau(0) = -1$.
2. τ *is concave.*

Proof.

1. Because $\mathbb{E}[|X(t)|^0] = c(0)t^{\tau(0)+1} = 0$ for all $t \in [0, T]$, it follows that $\tau(0) = -1$. Hence the scaling function always has intercept -1.

2. Let $\alpha \in [0, 1]$, and let $q_1, q_2 \in [0, T]$ such that $q_1 < q_2$. Setting $q = q_1(\alpha) + q_2(1 - \alpha)$, and defining $\mathfrak{p}(z)$ as the density function of $N(0, t)$, H́older's inequality gives

$$\begin{aligned}\mathbb{E}[|X(t)|^q] &= \mathbb{E}[|X(t)|^{q_1(\alpha)} \cdot |X(t)|^{q_2(1-\alpha)}] \\ &\leq \mathbb{E}[|X(t)|^{q_1}]^{\alpha} \cdot \mathbb{E}[|X(t)|^{q_2}]^{(1-\alpha)}.\end{aligned}$$

Multifractality gives

$$c(q)t^{\tau(q)+1} \leq (c(q_1)t^{\tau(q_1)+1})^{\alpha} \cdot (c(q_2)t^{\tau(q_2)+1})^{1-\alpha}.$$

Taking logarithms then gives

$$\ln c(q) + (\tau(q) + 1)\ln t \leq \alpha(\ln c(q_1) + (\tau(q_1) + 1)\ln t) + (1 - \alpha)(\ln c(q_2) + (\tau(q_2) + 1)\ln t).$$

Simplifying the inequality, we get

$$\ln c(q) + \tau(q)\ln t \leq (\alpha\tau(q_1) + (1 - \alpha)\tau(q_2))\ln t + (\alpha \ln c(q_1) + (1 - \alpha)\ln c(q_2))$$

Suppose t is sufficiently small such that $\ln t < 0$. Dividing both sides by $\ln t$ and letting $t \to 0$, we have

$$\tau(q) \geq \alpha\tau(q_1) + (1 - \alpha)\tau(q_2), \text{ which gives our result.}$$

□

B Images

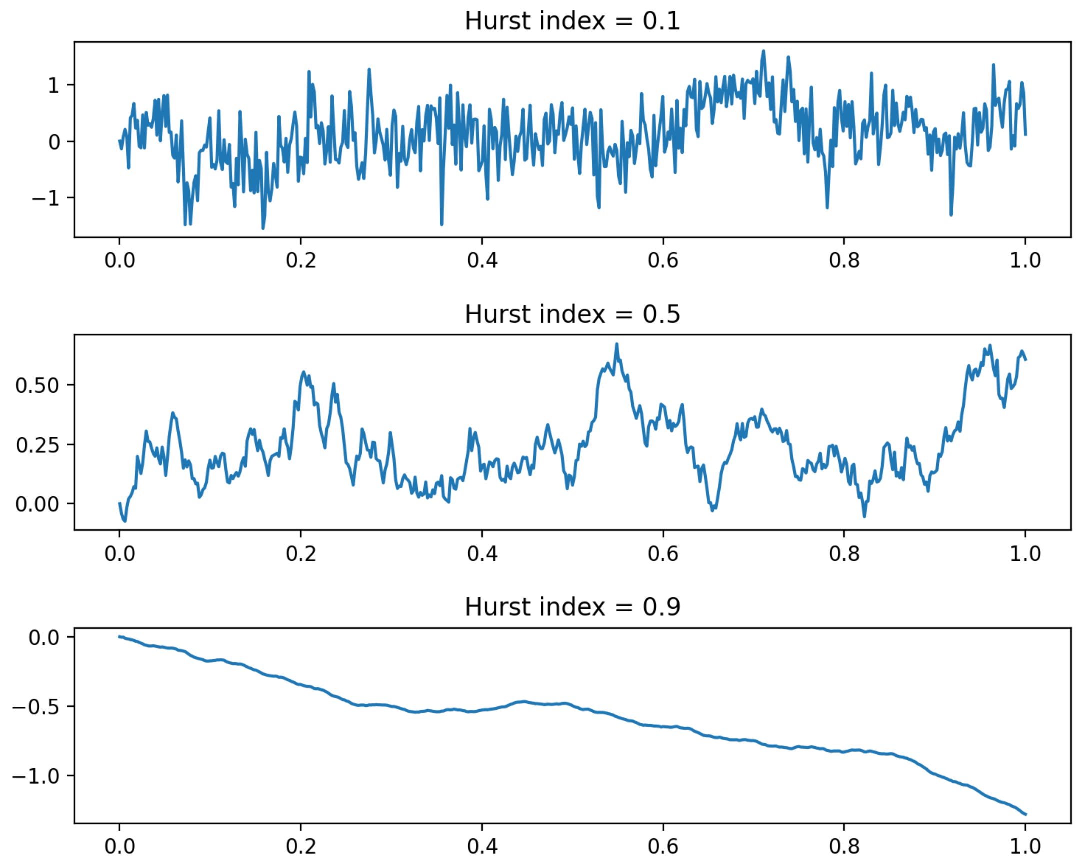

Figure 1: Fractional Brownian Motion with differing Hurst indices. As the Hurst index increases, the "fractalness" appears to decrease as Theorem 3.6 suggests. Furthermore, notice the persistence/anti-persistence trends, as noted in Proposition 3.4. These simulations were generated using crflynn's stochastic package.

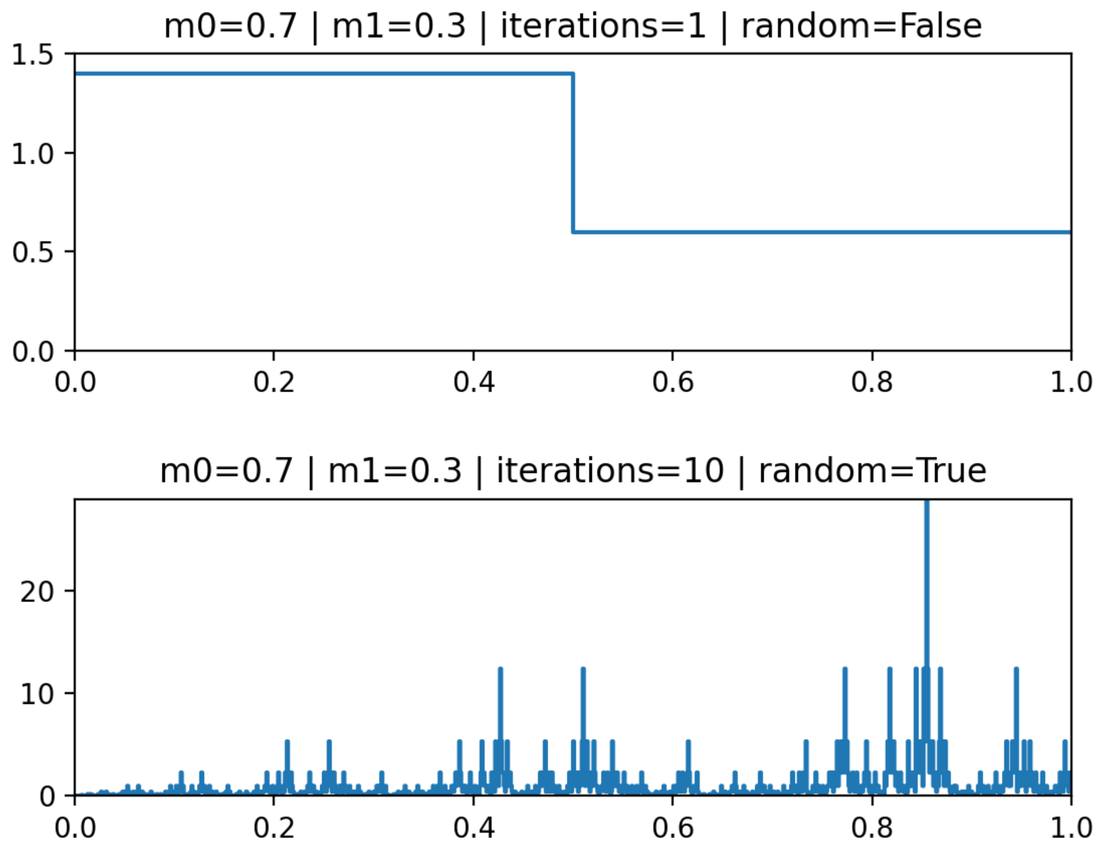

Figure 2: The binomial measure. The top shows μ_1 using $m_0 = 0.7$ and $m_1 = 0.3$. The bottom shows μ_{10} using the same weights, but with randomization of which weights go left or right. The generation code is shown in the code section of the appendix.

C Code

C.1 Code used to generate binomial measure

```
import matplotlib.pyplot as plt
import numpy as np

def split_binom():
  np.random.seed(3)

  m_0 = np.random.uniform(0,1,1)
  m_1 = 1 - m_0[0]

  return [m_0[0], m_1]

def random_binom_cascade(grid, values, iterations=3, key=3, random=True):
  grid_length = len(grid)
```

```
        value_length = len(values)

        if grid_length != value_length:
          raise ValueError("grid and value list lengths must match")

        if iterations == 0:
          updated_grid = grid + [1.0]
          updated_values = [i*(2**key) for i in values + [values[len(values) - 1]]]
          return updated_grid, updated_values

        else:
          new_grid = [i/(grid_length * 2) for i in range(grid_length * 2)]
          new_values = []

          m = split_binom()

          for orig_point in range(grid_length):
          choice = 0
          if random:
          choice = np.random.choice(2)

          left_multiplier = values[orig_point] * m[choice]
          right_multiplier = values[orig_point] * m[1-choice]

          new_values = new_values + [left_multiplier, right_multiplier]

          return random_binom_cascade(new_grid, new_values,
                          iterations=iterations-1,
                          key=key,
                          random=random)
```